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Secondary flow in an elastico-viscous fluid caused by rotational oscillations of a sphere. Part 1

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When the angular amplitude of oscillation of a sphere in an infinite mass of elastico-viscous fluid is fairly small, so that velocity and stress components may be expanded as power series in this amplitude, the purely periodic primary motion has associated with it a secondary flow which has a steady component as well as a component of double the primary frequency. An expression for the stream function of the steady secondary flow is obtained for all possible frequencies and the results are illustrated by considering in detail a particular fluid. It is shown that the streamline projections on a plane containing the axis of rotation are strongly dependent on the parameters measuring the elasticity of the fluid and on the frequency. The circulatory secondary flow can be in the opposite sense to that in a Newtonian fluid, in either the whole or part of the elastico-viscous fluid.

1. Introduction

Because of the non-linear nature of the equations of state of even the simplest elastico-viscous fluid, it is virtually impossible except when the fluid undergoes steady simple shearing, to obtain exact solutions of the equations of motion, and one must resort to approximate methods. In almost all published investigations to date the flow has been considered slow and the parameters measuring elastic properties of the fluid have been assumed small. As some of these materials are of great industrial importance it would be useful to study their flow under less restrictive conditions. Some progress can be made by considering motions in which the fluid is subjected to oscillations that are small, but not infinitesimal. Then it is not necessary to introduce restrictions on the magnitude of the Reynolds number describing the flow or on those of the parameters that measure the elasticity of the fluid.

A number of authors (for example, Burgers 1948; Oldroyd 1951; Walters 1960) have considered oscillatory flow of elastico-viscous fluids, but in each case the amplitude of oscillation was taken to be so small that all second-degree terms in the equations of motion could be neglected. In a previous paper (Frater 1964), the flow of a type of elastico-viscous fluid between torsionally oscillating disks was considered. It was found that a purely oscillatory motion of the disks produces a steady secondary flow, which for a certain critical range of frequencies shows a remarkable departure from the corresponding flow encountered in a Newtonian fluid. The rotational nature of the motion during each half-cycle gives rise to a normal-stress effect, comparable with the Weissenberg effect in steady flows.

The present paper treats the flow of an infinite mass of elastico-viscous fluid due to the oscillatory motion of a solid sphere about a fixed diameter. The corresponding flow in a Newtonian fluid has been given by Carrier & Di Prima (1956). These authors found that the secondary motion induced is such that fluid is repelled from the sphere at the equator and sucked in axially at the poles—a circulatory motion in planes containing the axis of rotation. The main purpose of their paper was to compute the correction to the oscillating viscous torque on the sphere when second-order terms are taken into account. We shall be principally concerned with the effect of the elasticity of the fluid on the nature of the secondary flow.

The idealized incompressible elastico-viscous fluid considered in the present paper has the following equations of state, relating the stress tensor S_{ik} and the rate-of-strain tensor $E_{ik} = \frac{1}{2}(U_{k,i} + U_{i,k})$:

$$S_{ik} = P_{ik} - Pg_{ik}, \tag{1}$$

$$P^{ik} + \lambda_1(\mathfrak{b}P^{ik}/\mathfrak{b}T) + \mu_0 P^j_j E^{ik} = 2\eta_0[E^{ik} + \lambda_2(\mathfrak{b}E^{ik}/\mathfrak{b}T)]. \tag{2}$$

Here U_i denotes the velocity vector, g_{ik} the metric tensor, P_{ik} the part of the stress tensor related to change of shape of a material element, and P an isotropic pressure; η_0 is a constant having the dimensions of viscosity (which can be identified with the limiting viscosity at vanishingly small constant rate of shear) and λ_1, λ_2, μ_0 are constants having the dimension of time. The derivative $\mathfrak{b}/\mathfrak{b}T$ is the convected time derivative (Oldroyd 1950) defined thus: if B^{ik} is any contravariant tensor,

$$\mathfrak{b}B^{ik}/\mathfrak{b}T \equiv \partial B^{ik}/\partial T + U^j B^{ik}_{\ ,j} + \Omega^i_{\ .m} B^{mk} + \Omega^k_{\ .m} B^{im} - E^i_m B^{mk} - E^k_m B^{im},$$

where $\Omega_{ik} = \frac{1}{2}(U_{k,i} - U_{i,k})$ is the vorticity tensor and a suffix following a comma denotes a covariant derivative; T is the time.

It has been shown (Oldroyd 1958) that the class of idealized fluids defined by equations (1) and (2) exhibit qualitatively most of the observed non-Newtonian features of some polymer solutions and other elastico-viscous fluids, provided the constants η_0, λ_1, λ_2 and μ_0 are chosen so that

$$\eta_0 > 0, \quad \lambda_1 > \lambda_2 \geqslant \tfrac{1}{9}\lambda_1 > 0, \quad \mu_0 > 0.$$

2. Equations of motion

We consider a solid sphere immersed in an infinite mass of elastico-viscous fluid, characterized by the set of equations (1) and (2), and suppose the sphere to be represented by $R = a$, in a spherical polar co-ordinate system (R, θ, ϕ) with origin at the centre of the sphere. The axis $\theta = 0$ is taken as the axis of symmetry for the whole motion. We shall suppose that the physical components of the velocity vector referred to these co-ordinates are U, V, W. If the sphere $R = a$ performs oscillations about the axis $\theta = 0$ with frequency $n/2\pi$ and angular amplitude Ω, the boundary conditions are

$$\left.\begin{aligned} &U = 0, \quad V = 0, \quad W = an\Omega \sin\theta\, e^{inT} \quad \text{on} \quad R = a, \\ &U = 0, \quad V = 0, \quad W = 0 \quad \text{as} \quad R \to \infty. \end{aligned}\right\} \tag{3}$$

The convention is adopted that real parts are to be understood whenever complex expressions are quoted for physical quantities.

Equations (1) and (2), together with the usual equations of motion and continuity, are first reduced to non-dimensional form by the following substitutions

$$R = ar, \quad T = n^{-1}t,$$
$$U = anu, \quad V = anv, \quad W = an\Omega w,$$
$$P_{(ik)} = n\eta_0\, p_{(ik)}, \quad P - \rho Rg\cos\theta = \rho a^2n^2p,$$
$$\lambda_2 = \sigma\lambda_1, \quad \mu_0 = \epsilon\lambda_1,$$

where $P_{(ik)}$ denotes the physical components of the partial stress tensor and σ, ϵ are clearly two positive dimensionless physical constants of the material. For convenience, we shall from this point represent physical components of the partial stress tensor by p_{rr}, $p_{r\phi}$, etc., omitting the brackets around suffixes. We then obtain the following set of ten equations relating six components of partial stress, three components of velocity and an isotropic pressure:

$$p_{rr} + S\left(\frac{\partial p_{rr}}{\partial t} + u\frac{\partial p_{rr}}{\partial r} + \frac{v}{r}\frac{\partial p_{rr}}{\partial \theta} - 2\frac{\partial u}{\partial r}p_{rr} - \frac{2}{r}\frac{\partial u}{\partial \theta}p_{r\theta}\right) + \epsilon S\, p_{jj}\frac{\partial u}{\partial r}$$
$$= 2\frac{\partial u}{\partial r} + 2\sigma S\left[\frac{\partial^2 u}{\partial t\,\partial r} + u\frac{\partial^2 u}{\partial r^2} + \frac{v\,\partial^2 u}{r\,\partial r\,\partial \theta} - 2\left(\frac{\partial u}{\partial r}\right)^2 - \frac{\partial u}{r\,\partial\theta}\left\{\frac{\partial u}{r\,\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}\right], \quad (4)$$

$$p_{\theta\theta} + S\left[\frac{\partial p_{\theta\theta}}{\partial t} + r^2u\frac{\partial}{\partial r}\left(\frac{p_{\theta\theta}}{r^2}\right) + \frac{v}{r}\frac{\partial p_{\theta\theta}}{\partial\theta} - 2r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)p_{r\theta} - \frac{2}{r}\frac{\partial v}{\partial\theta}p_{\theta\theta}\right] + \epsilon S\, p_{jj}\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}\right)$$
$$= 2\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}\right) + 2\sigma S\left[\frac{\partial}{\partial t}\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}\right) + r^2u\frac{\partial}{\partial r}\left(\frac{u}{r^3} + \frac{\partial v}{r^3\,\partial\theta}\right) + \frac{v}{r}\frac{\partial}{\partial\theta}\left(\frac{u}{r} + \frac{\partial v}{r\,\partial\theta}\right)\right.$$
$$\left. - \frac{2}{r}\frac{\partial v}{\partial\theta}\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}\right) - r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\left\{\frac{\partial u}{r\,\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}\right], \quad (5)$$

$$p_{\phi\phi} + S\left[\frac{\partial p_{\phi\phi}}{\partial t} + r^2u\frac{\partial}{\partial r}\left(\frac{p_{\phi\phi}}{r^2}\right) + v\sin^2\theta\frac{\partial}{r\,\partial\theta}\left(\frac{p_{\phi\phi}}{\sin^2\theta}\right) - 2r\Omega\frac{\partial}{\partial r}\left(\frac{w}{r}\right)p_{\phi r}\right.$$
$$\left. - 2\Omega\sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{w}{\sin\theta}\right)p_{\theta\phi}\right] + \epsilon S\left(\frac{u}{r} + \frac{v\cot\theta}{r}\right)p_{jj}$$
$$= 2\left(\frac{u}{r} + \frac{v\cot\theta}{r}\right) + 2\sigma S\left[\frac{\partial}{\partial t}\left(\frac{u}{r} + \frac{v\cot\theta}{r}\right) + r^2u\frac{\partial}{\partial r}\left(\frac{u}{r^3} + \frac{v\cot\theta}{r^3}\right)\right.$$
$$\left. + v\sin^2\theta\frac{\partial}{r\,\partial\theta}\left(\frac{u}{r\sin^2\theta} + \frac{v\cos\theta}{r\sin^3\theta}\right) - \Omega^2\left\{r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right\}^2 - \Omega^2\left\{\sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{w}{\sin\theta}\right)\right\}^2\right], \quad (6)$$

$$p_{\theta\phi} + S\left[\frac{\partial p_{\theta\phi}}{\partial t} + r^2u\frac{\partial}{\partial r}\left(\frac{p_{\theta\phi}}{r^2}\right) + v\sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{p_{\theta\phi}}{\sin\theta}\right) - \Omega r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)p_{r\theta} - r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)p_{\phi r}\right.$$
$$\left. - \Omega\sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{w}{\sin\theta}\right)p_{\theta\theta} - \frac{\partial v}{r\,\partial\theta}p_{\phi\phi}\right] + \tfrac{1}{2}\epsilon S\sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{w}{\sin\theta}\right)p_{jj}$$
$$= \Omega\left\{\sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{w}{\sin\theta}\right) + \sigma S\left[\frac{\partial}{\partial t}\sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{w}{\sin\theta}\right) + r^2u\frac{\partial}{\partial r}\left\{\sin\theta\frac{\partial}{r^3\,\partial\theta}\left(\frac{w}{\sin\theta}\right)\right\}\right.\right.$$
$$\left.\left. + v\sin\theta\frac{\partial^2}{r^2\,\partial\theta^2}\left(\frac{w}{\sin\theta}\right) - r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\left\{\frac{\partial u}{r\,\partial\theta} + 2r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\} - \sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{w}{\sin\theta}\right)\left\{2\frac{u}{r} + \frac{3}{r}\frac{\partial v}{\partial\theta}\right\}\right]\right\}, \quad (7)$$

$$
\begin{aligned}
p_{\phi r}+S\left[\frac{\partial p_{\phi r}}{\partial t}+ru\frac{\partial}{\partial r}\left(\frac{p_{\phi r}}{r}\right)+v\sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{p_{\phi r}}{\sin\theta}\right)-\Omega r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)p_{rr}\right. \\
\left.-\Omega\sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{w}{\sin\theta}\right)p_{\phi r}-\frac{\partial u}{r\,\partial\theta}p_{\theta\phi}-\frac{\partial u}{\partial r}p_{\phi r}\right]+\tfrac{1}{2}\epsilon Sr\frac{\partial}{\partial r}\left(\frac{w}{r}\right)p_{jj} \\
=\Omega\left\{r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)+\sigma S\left[\frac{\partial}{\partial t}r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)+ru\frac{\partial^2}{\partial r^2}\left(\frac{w}{r}\right)+v\sin\theta\frac{\partial}{\partial\theta}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right\}\right.\right. \\
\left.\left.-3r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\frac{\partial u}{\partial r}-\sin\theta\frac{\partial}{r\,\partial\theta}\left(\frac{w}{\sin\theta}\right)\left\{2\frac{\partial u}{r\,\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}\right]\right\},
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
p_{r\theta}+S\left[\frac{\partial p_{r\theta}}{\partial t}+ur\frac{\partial}{\partial r}\left(\frac{p_{r\theta}}{r}\right)+\frac{v}{r}\frac{\partial}{\partial\theta}p_{r\theta}-r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)p_{rr}-\left(\frac{\partial u}{\partial r}+\frac{\partial v}{r\,\partial\theta}\right)p_{r\theta}-\frac{\partial u}{r\,\partial\theta}p_{\theta\theta}\right] \\
+\tfrac{1}{2}\sigma S\left\{\frac{\partial u}{r\,\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}p_{jj} \\
=\frac{\partial u}{r\,\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\sigma S\left[\frac{\partial}{\partial t}\left\{\frac{\partial u}{r\,\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}+ru\frac{\partial}{\partial r}\left\{\frac{\partial u}{r^2\,\partial\theta}+\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}\right. \\
+v\left\{\frac{\partial^2 u}{r^2\,\partial\theta^2}+\frac{\partial^2}{\partial\theta\,\partial r}\left(\frac{v}{r}\right)\right\}-2r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\frac{\partial u}{\partial r}-2\frac{\partial u}{r\,\partial\theta}\left(\frac{u}{r}+\frac{\partial v}{r\,\partial\theta}\right) \\
\left.-\left(\frac{\partial u}{\partial r}+\frac{\partial v}{r\,\partial\theta}\right)\left\{\frac{\partial u}{r\,\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\}\right],
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\theta}-\frac{v^2+\Omega^2w^2}{r} \\
=-\frac{\partial p}{\partial r}+\frac{1}{R_0}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2p_{rr})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta p_{r\theta})-\frac{p_{\theta\theta}+p_{\phi\phi}}{r}\right],
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial t}+\frac{u}{r}\frac{\partial}{\partial r}(rv)+\frac{v}{r}\frac{\partial v}{\partial\theta}-\frac{\Omega^2w^2\cot\theta}{r} \\
=-\frac{\partial p}{r\,\partial\theta}+\frac{1}{R_0}\left[\frac{1}{r^3}\frac{\partial}{\partial r}(r^3p_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta p_{\theta\theta})-\frac{\cot\theta}{r}p_{\phi\phi}\right],
\end{aligned}
\tag{11}
$$

$$
\Omega\left[\frac{\partial w}{\partial t}+\frac{u}{r}\frac{\partial}{\partial r}(rw)+\frac{v}{r\sin\theta}\frac{\partial}{\partial\theta}(w\sin\theta)\right]=\frac{1}{R_0}\left[\frac{1}{r^3}\frac{\partial}{\partial r}(r^3p_{\phi r})+\frac{1}{r\sin^2\theta}\frac{\partial}{\partial\theta}(\sin^2\theta p_{\theta\phi})\right],
\tag{12}
$$

$$
\frac{\partial}{\partial r}(r^2u\sin\theta)+\frac{\partial}{\partial\theta}(rv\sin\theta)=0,
\tag{13}
$$

where $R_0=\rho na^2/\eta_0$ is a Reynolds number for the flow, $p_{jj}=p_{rr}+p_{\theta\theta}+p_{\phi\phi}$, and $S=\lambda_1 n$ can be thought of as a dimensionless measure of the 'memory' of the elastico-viscous fluid, based on the use of the period of oscillation as the natural unit of time. The associated boundary conditions are

$$
\left.
\begin{array}{llll}
u=0, & v=0, & w=\sin\theta\,e^{it} & \text{on} \quad r=1, \\
u=0, & v=0, & w=0 & \text{as} \quad r\to\infty.
\end{array}
\right\}
\tag{14}
$$

3. Solution of the equations

In order to obtain an approximate solution of the above equations, it is now assumed that Ω is fairly small, so that we may expand certain quantities as power series in Ω. Then from the form of the equations (4)–(13) and the boundary conditions (14), we can write

$$\begin{aligned}
w &= f(r)\sin\theta\, e^{it} + \Omega^2 f_1(r,\theta,t) + \dots,\\
u &= \Omega^2 g(r,\theta,t) + \dots, \quad v = \Omega^2 h(r,\theta,t) + \dots,\\
p_{\phi r} &= \Omega F(r)\sin\theta\, e^{it} + \Omega^3 F_1(r,\theta,t) + \dots,\\
p_{rr} &= \Omega^2 G(r,\theta,t) + \dots, \quad p_{\theta\theta} = \Omega^2 H(r,\theta,t) + \dots,\\
p_{\phi\phi} &= \Omega^2 K(r,\theta,t) + \dots, \quad p_{r\theta} = \Omega^2 L(r,\theta,t) + \dots,\\
p_{\theta\phi} &= \Omega^3 M(r,\theta,t) + \dots, \quad p = \Omega^2 N(r,\theta,t) + \dots.
\end{aligned}$$

If these expressions are substituted in equations (4)–(13) and the boundary conditions (14), and coefficients of Ω, Ω^2, etc., are equated, the following system of linear partial differential equations is obtained:

$$G + S\frac{\partial G}{\partial t} = 2\left(1+\sigma S\frac{\partial}{\partial t}\right)\frac{\partial g}{\partial r}, \tag{15}$$

$$H + S\frac{\partial H}{\partial t} = 2\left(1+\sigma S\frac{\partial}{\partial t}\right)\left(\frac{g}{r}+\frac{1}{r}\frac{\partial h}{\partial \theta}\right), \tag{16}$$

$$K + S\frac{\partial K}{\partial t} = 2\left(1+\sigma S\frac{\partial}{\partial t}\right)\left(\frac{g}{r}+\frac{h\cot\theta}{r}\right) + 2S\sin^2\theta\left[r\frac{d}{dr}\left(\frac{f}{r}\right)e^{it}F\,e^{it} - \left\{r\frac{d}{dr}\left(\frac{f}{r}\right)e^{it}\right\}^2\right], \tag{17}$$

$$(1+iS)\,F = (1+i\sigma S)\,r\frac{d}{dr}\left(\frac{f}{r}\right), \tag{18}$$

$$L + S\frac{\partial L}{\partial t} = \left(1+\sigma S\frac{\partial}{\partial t}\right)\left[\frac{1}{r}\frac{\partial g}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{h}{r}\right)\right], \tag{19}$$

$$\frac{\partial g}{\partial t} - \frac{\sin^2\theta}{r}(f e^{it})^2 = -\frac{\partial N}{\partial r} + \frac{1}{R_0}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 G) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(L\sin\theta) - \frac{H+K}{r}\right], \tag{20}$$

$$\frac{\partial h}{\partial t} - \frac{\sin^2\theta}{r}\cot\theta(f e^{it})^2 = -\frac{\partial N}{r\partial\theta} + \frac{1}{R_0}\left[\frac{1}{r}\frac{\partial}{\partial r}(r^3 L) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H\sin\theta) - \frac{\cot\theta}{r}K\right], \tag{21}$$

$$if = \frac{1}{R_0}\left[\frac{1}{r^3}\frac{d}{dr}(r^3 F)\right], \tag{22}$$

$$\frac{\partial}{\partial r}(r^2 g\sin\theta) + \frac{\partial}{\partial\theta}(rh\sin\theta) = 0. \tag{23}$$

The boundary conditions become

$$\left.\begin{aligned}
f = 1, \quad g = 0, \quad h = 0 \quad \text{on} \quad r = 1,\\
f = 0, \quad g = 0, \quad h = 0 \quad \text{as} \quad r \to \infty.
\end{aligned}\right\} \tag{24}$$

The primary motion

Eliminating F from equations (18) and (22) gives the following ordinary differential equation for f:

$$r^2 f'' + 2rf' - (\alpha^2 R_0 + 2) f = 0, \tag{25}$$

where
$$\alpha^2 = i\{(1+iS)/(1+i\sigma S)\}, \tag{26}$$

and a prime denotes differentiation with respect to r. The solution of equation (25) satisfying the boundary conditions (24) is

$$f(r) = \frac{1+\alpha R_0^{\frac{1}{2}} r}{(1+\alpha R_0^{\frac{1}{2}}) r^2} \exp\left[-\alpha R_0^{\frac{1}{2}}(r-1)\right]. \tag{27}$$

The corresponding expression for $F(r)$ is

$$F(r) = -\frac{i}{\alpha^2}\left[\frac{3+3\alpha R_0^{\frac{1}{2}} r + \alpha^2 R_0 r^2}{(1+\alpha R_0^{\frac{1}{2}}) r^3}\right] \exp\left[-\alpha R_0^{\frac{1}{2}}(r-1)\right]. \tag{28}$$

We suppose the real and imaginary parts of $\alpha R_0^{\frac{1}{2}}$ are b and q, so that $f(r)$ and $F(r)$ can be written in the forms

$$f(r) = \left(\frac{1+2br+r^2(b^2+q^2)}{r^4(1+2b+b^2+q^2)}\right)^{\frac{1}{2}} \exp\left[-(r-1)b - i\{(r-1)q - \eta_1(r)\}\right], \tag{29}$$

$$F(r) = -\frac{iR_0 e^{-2i\chi}}{b^2+q^2}\left(\frac{(3qr+2bqr^2)^2 + \{3+3br+(b^2-q^2)r^2\}^2}{r^6(1+2b+b^2+q^2)}\right)^{\frac{1}{2}}$$
$$\times \exp\left[-(r-1)b - i\{(r-1)q - \eta_2(r)\}\right], \tag{30}$$

where
$$\chi = \tfrac{1}{4}\pi + \tfrac{1}{2}\tan^{-1} S - \tfrac{1}{2}\tan^{-1}\sigma S, \tag{31}$$

$$\eta_1(r) = \tan^{-1}\{q(r-1)/(1+b+br+b^2 r+q^2 r)\}, \tag{32}$$

$$\eta_2(r) = \tan^{-1}\left\{\frac{(1+b)(3qr+2bqr^2) - b\{3+3br+(b^2-q^2)r^2\}}{(1+b)\{3+3br+(b^2-q^2)r^2\} + b(3qr+2bqr^2)}\right\}. \tag{33}$$

The secondary motion

From the fact that, if z_1 and z_2 are any two complex numbers,*

$$\operatorname{Re}(z_1)\operatorname{Re}(z_2) = \tfrac{1}{2}\operatorname{Re}(z_1\bar{z}_2) + \tfrac{1}{2}\operatorname{Re}(z_1 z_2),$$

we see that the quadratic non-homogeneous terms in equations (17), (20) and (21) will consist of a time-independent term plus a term proportional to $\exp(2it)$. Hence we will expect g, h, G, H, K and N to be given by

$$g = g_1 + g_2 e^{2it}, \quad h = h_1 + h_2 e^{2it}, \quad G = G_1 + G_2 e^{2it},$$
$$H = H_1 + H_2 e^{2it}, \quad K = K_1 + K_2 e^{2it}, \quad N = N_1 + N_2 e^{2it}.$$

* $\bar{z}$ denotes the complex conjugate of z.

Substituting these expressions into equations (15)–(23) and equating time-independent parts, we obtain the following system of linear partial differential equations:

$$G_1 = 2(\partial g_1/\partial r), \tag{34}$$

$$H_1 = 2\left(\frac{g_1}{r}+\frac{1}{r}\frac{\partial h_1}{\partial \theta}\right), \tag{35}$$

$$K_1 = 2\left(\frac{g_1}{r}+\frac{h_1 \cot\theta}{r}\right)+S\sin^2\theta\left\{r\frac{d}{dr}\left(\frac{f}{r}\right)\bar{F}-\sigma r^2\left[\frac{d}{dr}\left(\frac{f}{r}\right)\frac{d}{dr}\left(\frac{\bar{f}}{r}\right)\right]\right\}, \tag{36}$$

$$L_1 = \frac{1}{r}\frac{\partial g_1}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{h_1}{r}\right), \tag{37}$$

$$-\sin^2\theta\frac{f\bar{f}}{2r} = -\frac{\partial N_1}{\partial r}+\frac{1}{R_0}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2G_1)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(L_1\sin\theta)-\frac{H_1+K_1}{r}\right\}, \tag{38}$$

$$-\sin^2\theta\cot\theta\frac{f\bar{f}}{2r} = -\frac{\partial N_1}{r\,\partial\theta}+\frac{1}{R_0}\left\{\frac{1}{r^3}\frac{\partial}{\partial r}(r^3L_1)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H_1\sin\theta)-\frac{\cot\theta}{r}K_1\right\}, \tag{39}$$

$$\partial(r^2 g_1\sin\theta)/\partial r+\partial(rh_1\sin\theta)/\partial\theta = 0. \tag{40}$$

The associated boundary conditions are

$$\left.\begin{aligned} g_1 = 0,\quad h_1 = 0 \quad \text{on} \quad r = 1,\\ g_1 = 0,\quad h_1 = 0 \quad \text{as} \quad r\to\infty.\end{aligned}\right\} \tag{41}$$

From equation (40) it is seen that a stream function $\psi(r,\theta)$ exists such that

$$g_1 = \frac{1}{r\sin\theta}\frac{\partial(\psi\sin\theta)}{\partial\theta},\quad h_1 = -\frac{1}{r}\frac{\partial(r\psi)}{\partial r}. \tag{42}$$

Eliminating N_1 from equations (38) and (39) and substituting for G_1, L_1, H_1 and K_1, we obtain the following equation for ψ

$$\Delta^2\psi = R_0\xi(r)\sin 2\theta, \tag{43}$$

where
$$\Delta y \equiv \frac{1}{r}\left\{\frac{\partial^2(ry)}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial(y\sin\theta)}{\partial\theta}\right]\right\},$$

and
$$\xi(r) = \frac{1}{r}\left[\frac{1}{4}\frac{d}{dr}(f\bar{f})-\frac{1}{2}\frac{f\bar{f}}{r}+\frac{S}{R_0}\left\{\frac{d}{dr}\left(\frac{f}{r}\right)\bar{F}-\sigma r\left|\frac{d}{dr}\left(\frac{f}{r}\right)\right|^2\right\}\right.$$
$$\left.-\frac{1}{2}\frac{S}{R_0}\frac{d}{dr}\left\{r\frac{d}{dr}\left(\frac{f}{r}\right)\bar{F}-\sigma\left|r\frac{d}{dr}\left(\frac{f}{r}\right)\right|^2\right\}\right].$$

The form of equation (43) suggests that ψ will be given by

$$\psi(r,\theta) = \Psi(r)\sin 2\theta. \tag{44}$$

Substitution of this expression into equation (43) gives the following linear ordinary differential equation for $\Psi(r)$:

$$\Delta_1^2\Psi = R_0\xi(r), \tag{45}$$

where
$$\Delta_1 y \equiv \frac{1}{r}\left[\frac{d^2(ry)}{dr^2}-6\frac{y}{r}\right].$$

This equation must be solved with the boundary conditions

$$\left.\begin{aligned}&\text{(i)}\quad \Psi = 0,\quad \Psi' = 0 \quad \text{on} \quad r = 1,\\ &\text{(ii)}\quad \Psi = 0,\quad \Psi' = 0 \quad \text{as} \quad r \to \infty.\end{aligned}\right\} \tag{46}$$

The general solution of equation (45) may be obtained by the usual method of variation of parameters. After some straightforward (but rather laborious) algebra we find

$$\frac{\Psi(r)}{R_0} = \frac{1}{30r}\int_{C_1}^{r} \xi(s)s^4 ds - \frac{1}{70r^3}\int_{C_2}^{r} \xi(s)s^6 ds - \frac{r^2}{30}\int_{C_3}^{r} \xi(s)\, s\, ds + \frac{r^4}{70}\int_{C_4}^{r} \xi(s)s^{-1} ds, \tag{47}$$

where C_1, C_2, C_3 and C_4 are four arbitrary constants.

Substituting the values of f and $\bar{F}$, etc., into the expression for $\xi(r)$ we have

$$\xi(r) = -\delta^{-1}\left(\frac{A_1}{r^8} + \frac{A_2}{r^7} + \frac{A_3}{r^6} + \frac{A_4}{r^5} + \frac{A_5}{r^4} + \frac{A_6}{r^3}\right) e^{-k(r-1)},$$

where $\quad A_1 = 36\beta,\quad A_2 = 72b\beta,\quad A_3 = 9(7b^2+q^2)\beta + \tfrac{3}{2},$

$$A_4 = 6b(5b^2+3q^2)\beta + 3b,\quad A_5 = 2(b^2+q^2)(4b^2+q^2)\beta + (2b^2+q^2),$$

$$A_6 = q(b^2+q^2)^2\beta + \tfrac{1}{2}q(b^2+q^2),$$

$$\beta = \frac{\sigma S}{R_0} - \frac{S\sin 2\chi}{b^2+q^2},\quad \delta = b^2+q^2+2b+1,\quad k = 2b.$$

Substituting this value of $\xi(r)$ into equation (47), we find that the solution of equation (45) satisfying the boundary conditions (46) is given by

$$\frac{\delta}{R_0}\Psi(r) = \left(\frac{B_1}{r^4} + \frac{B_2}{r^3} + \frac{B_3}{r^2} + \frac{B_4}{r} + B_5 + B_6 r\right) e^{-k(r-1)} + B_7 r^2 e^k \int_r^\infty \frac{e^{-ks}}{s} ds + \frac{B_8}{r} + \frac{B_9}{r^3}, \tag{48}$$

where

$$B_1 = -\tfrac{1}{4}\beta,\quad B_2 = -\frac{\beta}{80b^3}(5b^4+14b^2q^2+q^4) - \frac{1}{160b^3}(5b^2+q^2),$$

$$B_3 = \frac{q^2\gamma}{80b^2},\quad B_4 = -\frac{q^2\gamma}{120b},\quad B_5 = \frac{q^2\gamma}{120},$$

$$B_6 = -\frac{q^2b\gamma}{60},\quad B_7 = \frac{q^2b^2\gamma}{30},\quad \gamma = 2(b^2-q^2)\beta - 1,$$

$$2B_8 = -\frac{\beta}{40}\left\{10 + 20b + \frac{1}{b^2}(5b^2+14b^2q^2+q^4)\right\} - \frac{1}{80b^2}(5b^2+q^2) + \gamma q^2\left\{-\frac{1}{80b^2} + \frac{1}{24b} - \frac{1}{24} + \frac{b}{12} - \frac{b^2}{6}e^k\int_1^\infty \frac{e^{-ks}}{s} ds\right\},$$

$$2B_9 = \frac{\beta}{40}\left\{30 + 20b + \frac{1}{b^3}(1+b)(5b^4+14b^2q^2+q^4)\right\} + \frac{1}{80b^3}(5b^2+q^2)(1+b) + \gamma q^2\left\{-\frac{1}{80b^2} - \frac{1}{40b} + \frac{1}{40} - \frac{b}{20} + \frac{b^2}{10}e^k\int_1^\infty \frac{e^{-ks}}{s} ds\right\}.$$

4. Discussion of results

As equation (48) stands, it is too complicated for any general deductions to be made about the behaviour of the flow. It is, however, possible to simplify this equation considerably if we make the assumption that the Reynolds number R_0 is large. This restriction will not be serious because, if there are any effects due to the elasticity of the fluid, these are bound to be present at fairly large Reynolds number. For large values of R_0, we may use the asymptotic expansion for $\int_r^\infty \frac{e^{-ks}}{s}\,ds$, as follows:

$$\int_r^\infty \frac{e^{-ks}}{s}\,ds \sim \left(\frac{1}{kr}-\frac{1}{k^2r^2}+\frac{2!}{k^3r^3}-\frac{3!}{k^4r^4}+\ldots\right)e^{-kr}.$$

Substitution of this expression into equation (48) gives the following asymptotic form for $\Psi(r)$:

$$\frac{\delta}{R_0}\Psi(r) \sim \frac{B_8'}{r}+\frac{B_9'}{r^3}+\left\{\frac{B_2'}{r^3}+\frac{B_1'}{r^4}+q^2\gamma\left(\frac{3}{16b^5r^5}-\frac{21}{32b^6r^6}+\frac{21}{8b^7r^7}-\frac{189}{16b^8r^8}\right)\right\}e^{-k(r-1)}, \tag{49}$$

where B_8' and B_9' are the asymptotic forms of B_8 and B_9, given by

$$2B_8' = -\frac{\beta}{40}\left\{10+20b+\frac{1}{b^2}(5b^4+14b^2q^2+q^4)\right\}-\frac{1}{80b^2}(5b^2+q^2)$$
$$+\gamma q^2\left(\frac{1}{20b^2}-\frac{1}{8b^3}+\frac{5}{16b^4}-\frac{15}{16b^5}+\frac{105}{32b^6}-\frac{105}{8b^7}+\frac{945}{16b^8}\right),$$

$$2B_9' = \frac{\beta}{40}\left\{30+20b+\frac{1}{b^3}(1+b)(5b^4+14b^2q^2+q^4)\right\}+\frac{1}{80b^3}(5b^2+q^2)(1+b)$$
$$+\gamma q^2\left(-\frac{1}{20b^2}+\frac{3}{40b^3}-\frac{3}{16b^4}+\frac{9}{16b^5}-\frac{63}{32b^6}+\frac{63}{8b^7}-\frac{567}{16b^8}\right),$$

and B_1' and B_2' are given by

$$B_1' = \frac{1}{16b^4}\{q^2-2\beta(2b^4+q^2b^2-q^4)\},\quad B_2' = -\frac{5}{160b^3}(b^2+q^2)\{2(b^2+q^2)\beta+1\}.$$

In these expressions we have neglected terms of order b^{-7} compared with unity. Remembering that $b = |\alpha|R_0^{\frac{1}{2}}\cos\chi$, and $|\alpha| \geqslant 1$, this is equivalent to taking $R_0^{\frac{7}{2}} \gg 1$.

For the purpose of numerical illustration, we now focus attention on one particular fluid (i.e. a fluid with given values of η_0, ρ, λ_1 and λ_2) and study the flow for different values of the frequency n. The two parameters R_0 and S are then each proportional to the frequency. The dimensionless parameter σ $(=\lambda_2/\lambda_1)$ is chosen to have the value $\frac{1}{4}$, and R_0/S $(=\rho a^2/\eta_0\lambda_1)$ the value 50. The calculations have been carried out with the aid of an IBM 1620 computer.

It is found that the behaviour of the secondary flow is strongly dependent on the frequency. We illustrate this dependence by constructing the projections of the streamlines of the secondary flow on any plane containing the axis of rotation. The projection of any streamline is represented by $r\psi \sin\theta =$ constant, i.e.

$$r\Psi(r)\sin\theta\sin 2\theta = \text{constant}. \tag{50}$$

Quite different forms of projection are obtained for values of R_0 lying in the approximate ranges: (i) $0 < R_0 < 36$, (ii) $36 < R_0 < 50$, (iii) $50 < R_0 < 244$, (iv) $244 < R_0 < 245$, (v) $245 < R_0$. The projections of the streamlines (which are not plotted at equally spaced values of the stream function) corresponding to each of these ranges of R_0 are now discussed. We need only consider one quadrant $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ because of symmetry.

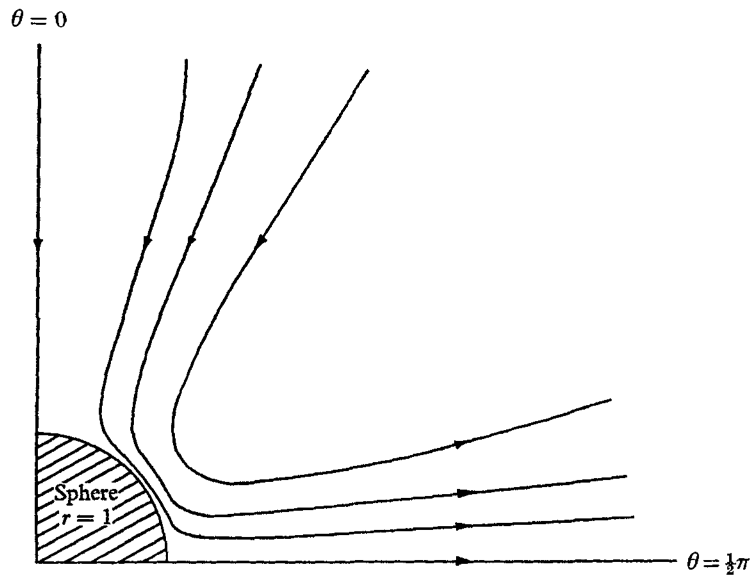

FIGURE 1. The projections of typical streamlines on a plane containing the axis of rotation when $\sigma = \frac{1}{4}$, $R_0 = 25$, $S = 0{\cdot}5$.

Case (i): $0 < R_0 < 36$

In this case, it is found that the function $r\Psi(r)$ is negative for all values of r, except $r = 1$ and $r = \infty$, where it vanishes. The flow pattern is then similar to that for a Newtonian fluid (corresponding to $\sigma = 1$); the fluid recedes from the sphere at the equator and approaches it at the poles. The projections of streamlines when $R_0 = 25$ are shown in figure 1.

Case (ii): $36 < R_0 < 50$

In this case, it is found that the function $r\Psi(r)$ changes sign for a finite value r' of r, defined by the equation $r'\Psi(r') = 0$ (different values of r' arising for different values of R_0). Inside the sphere $r = r'$, the projection curves are closed and fluid recedes from the sphere at the equator and approaches it again at the poles. Outside this sphere the curves are open and fluid recedes from the sphere at the poles and approaches it at the equator.

On the spherical surface $r = r'$, the radial velocity vanishes, although the component of velocity in the θ-direction does not. Hence it follows that particles of fluid which are initially inside (or outside) this sphere will remain so during the motion. The projections of streamlines in the cases $R_0 = 37$ and 38 are shown in figures 2(*a*) and (*b*).

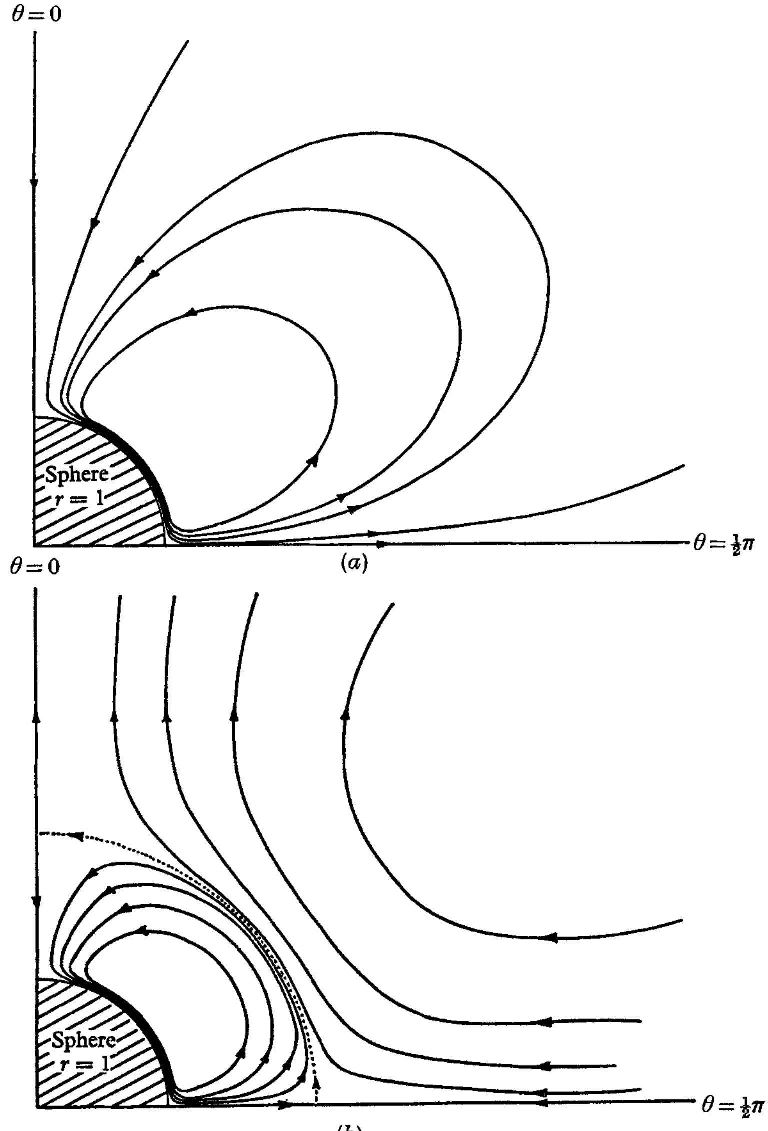

FIGURE 2. The projections of typical streamlines on a plane containing the axis of rotation when $\sigma = \frac{1}{4}$, (*a*) $R_0 = 37$, $S = 0{\cdot}74$, (*b*) $R_0 = 38$, $S = 0{\cdot}76$.

Case (iii): $50 < R_0 < 244$

In this case, it is found that the function $r\Psi'(r)$ is positive for all values of r lying in the range $0 < r < \infty$. The direction of flow is reversed compared with that in case (i); the fluid recedes from the sphere at the poles and approaches it at the equator. The projections of the streamlines for $R_0 = 100$ are shown in figure 3.

Case (iv): $244 < R_0 < 245$

The projections of the streamlines for $R_0 = 244{\cdot}1$ are shown in figure 4. As in case (ii), it is found that the function $r\Psi(r)$ changes sign for a finite value r'' of r (depending on the Reynolds number), given by $r''\Psi(r'') = 0$. As before, we find that the projections are closed curves inside $r = r''$, but now fluid recedes from the sphere at the poles and approaches it again at the equator. Outside the surface $r = r''$, the projections of streamlines are open and fluid recedes from the sphere at the equator and approaches it at the poles.

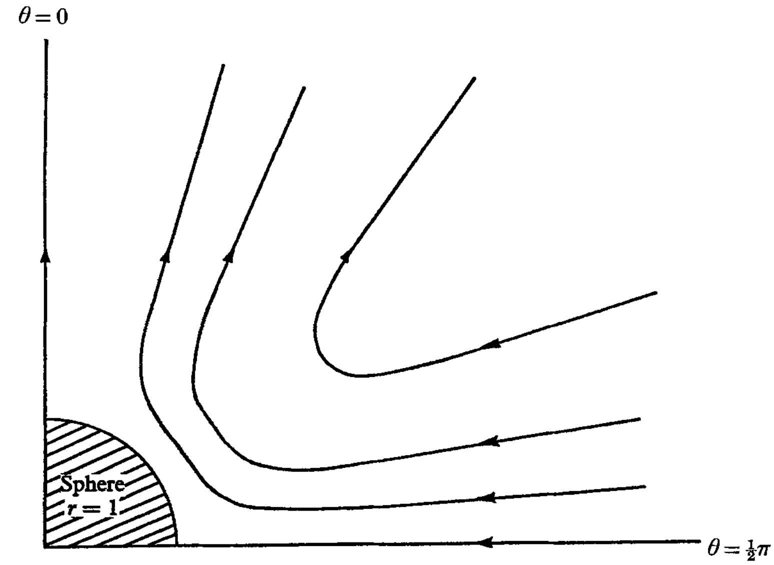

FIGURE 3. The projections of typical streamlines on a plane containing the axis of rotation when $\sigma = \frac{1}{4}$, $R_0 = 100$, $S = 2$.

Case (v): $245 < R_0$

In this case, the function $r\psi(r)$ is again negative for all values of r lying in the range $0 < r < \infty$. The flow pattern is very similar to that in case (i).

We conclude that, for σ given in a certain range (which includes $\sigma = \frac{1}{4}$ when $\rho a^2/(n_0\lambda_1) = 50$), there is a critical range of frequency in which the direction of the steady secondary flow is reversed compared with that in a Newtonian fluid. The predicted reversal phenomenon may be thought of as analogous to the Weissenberg effect in certain steady flows and it clearly arises because of the rotational nature of the motion of the sphere. The extra tension along the streamlines induced by shearing remains non-negative throughout each period of oscillation and has the effect of squeezing the fluid in certain regions towards the axis of symmetry. The relative importance of this effect is very dependent on the frequency. It is unimportant in case (i) and becomes comparable with the centrifugal-force effect when the frequency lies in the range considered in case (ii). At first it only affects the flow at large distances from the sphere but as the frequency is increased we see from figures 2(*a*) and (*b*) that it becomes important everywhere except in regions close to the sphere. For higher frequencies figure 3 shows that this effect dominates the centrifugal-force effect in the whole of the elastico-viscous fluid. As the frequency is increased further, however, the elastic effect becomes slightly less marked and at large distances from the sphere the centrifugal-force effect again becomes the more important. For yet larger frequencies the elastic effect is significant only in regions close to the sphere (see figure 4), and for larger still it becomes negligible everywhere. For further details of the dependence of the elastic effect on frequency see Frater (1964).

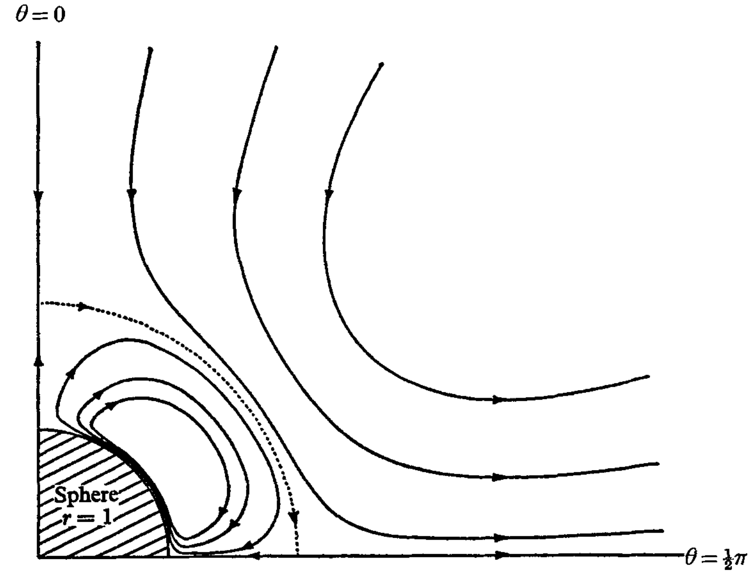

FIGURE 4. The projections of typical streamlines on a plane containing the axis of rotation when $\sigma = \frac{1}{4}$, $R_0 = 244{\cdot}1$, $S = 4{\cdot}882$.

Finally, we note that there is a certain similarity between the secondary flow found in the present oscillatory motion and the secondary flow caused by slow steady rotation of a solid sphere in an infinite mass of another type of elastico-viscous fluid as reported by Thomas & Walters (1964).

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